CORM 2011 NIST Measurement Uncertainty Workshop
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Measurement Uncertainty Evaluation
Where do I start?

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Outline

1. Terms and Concepts
2. Uncertainty Analysis Procedure
3. Example of Uncertainty Analysis
4. Summary & Conclusions

References

Measurement Uncertainty

Measurement result is complete only when a quantitative estimate of the uncertainty in the measurement is stated.

The final corrected result is the best estimate of the value of the quantity intended to be measured. The "true value" of the measurand is unknown.

Formal definition

Uncertainty of measurement is a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand.

"Expressed as a standard deviation (u)"

Why do you need an uncertainty budget?

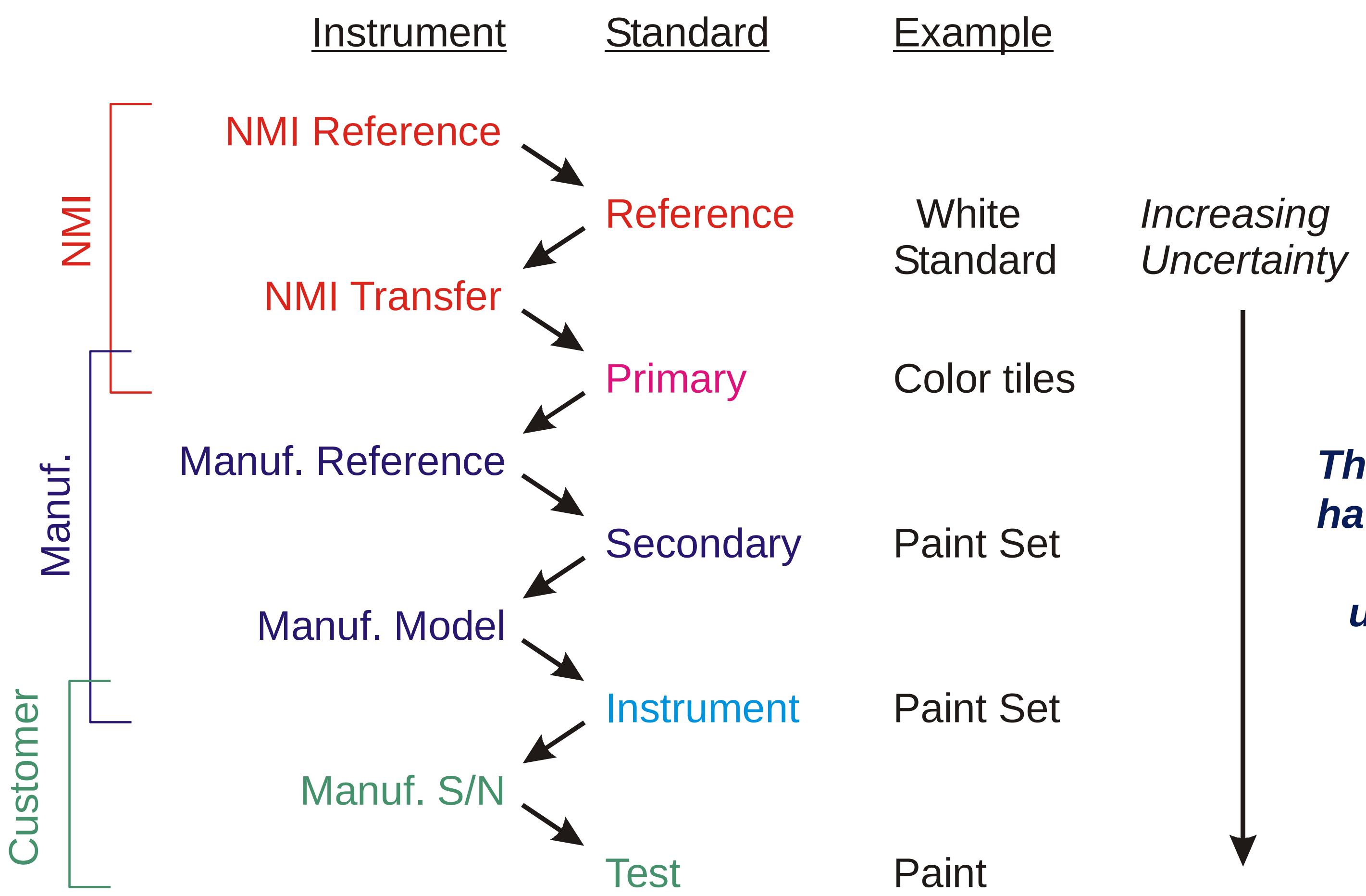
Traceability- "Property of the result of a measurement or the value of a standard whereby it can be related to stated references, usually national or international standards, through an unbroken chain of comparisons, all having stated uncertainties."

ISO *International Vocabulary of Basic and General Terms in Metrology*, 2nd ed., 1993, definition 6.10

Uncertainty budget will enable one to identify the dominant terms in the uncertainties to reduce those terms.

Traceability Chain

unbroken chain of comparisons, all having stated uncertainties



The more two chains have in common, the lower the uncertainty of the comparison

Background

1980: Development of international consensus on uniform approach
1981: Recommendation by International Committee for Weights and Measures (CIPM)
1986: Reaffirmation of CIPM recommendation
1993: Comprehensive reference document by working group

International Bureau of Weights and Measures (BIPM)
International Electrotechnical Commission (IEC)
International Organization for Standardization (ISO)
International Organization for Legal Metrology (OIML)

NIST adaptation of CIPM approach, policy and guidelines (1993)
(Taylor & Kuyatt, NIST Technical Note 1297, 1994)

Repeatability and Reproducibility

Closeness of agreement between the results of successive measurements of the same measurand carried out

Repeatability	Reproducibility
under same conditions of measurement	under changed conditions of measurement
Same	**Different**
principle	principle
method	method
observer	observer
location	location
instrument	instrument
time	time

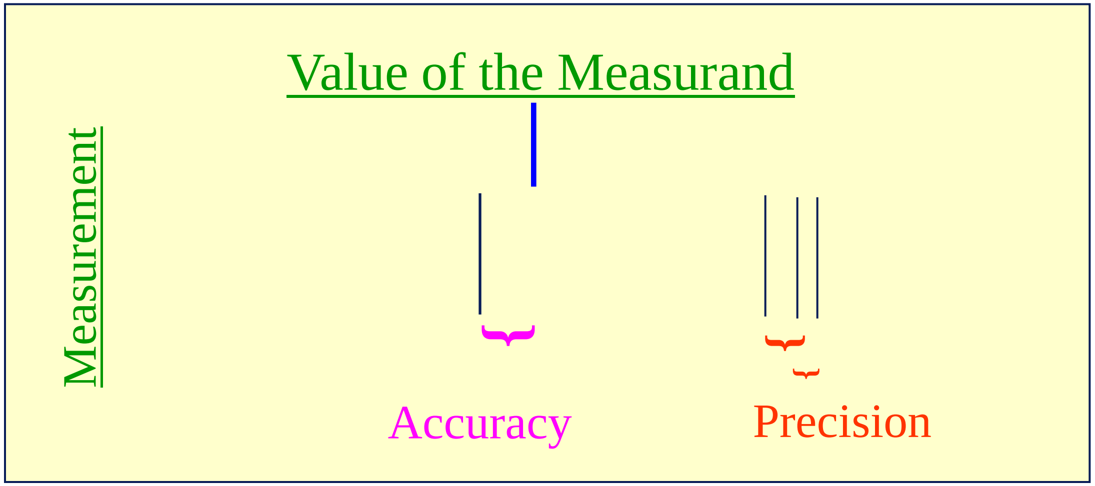
Accuracy and Precision

Accuracy

Closeness of agreement between the result of a measurement and the value of the measurand.

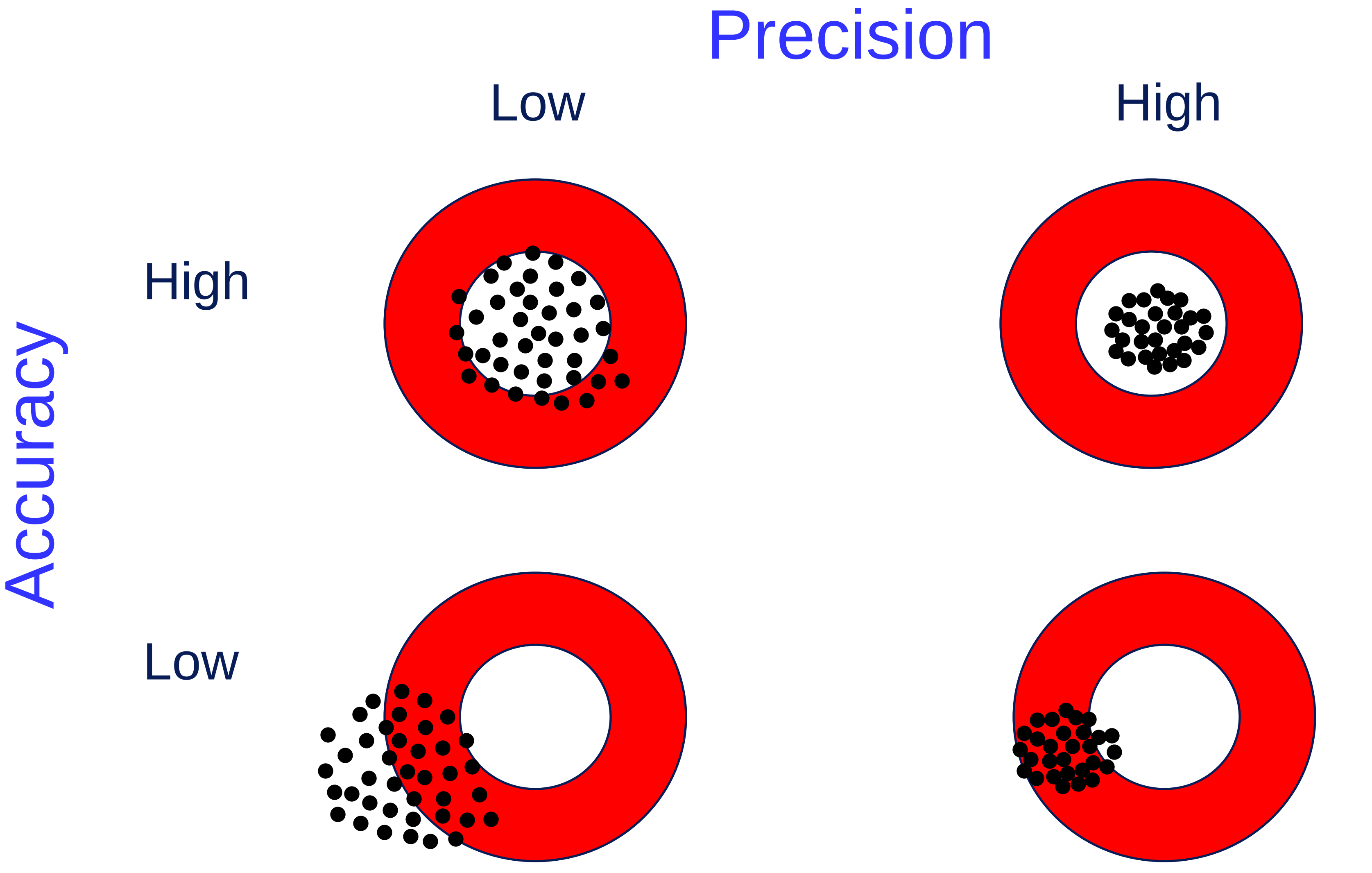
Precision

Closeness of agreement between the results of measurements of the same measurand.



Note: The ISO Guide to Uncertainty in Measurements (GUM) discourages the use of the terms, but are still used and confused in common usage.

Accuracy and Precision - Example



Error of Measurement

Result of a measurement minus the value of the measurand.
(Sum of random and systematic errors)

Random error

Result of a measurement
minus
the mean that would result from an infinite number of measurements of the same measurand carried out under repeatability conditions

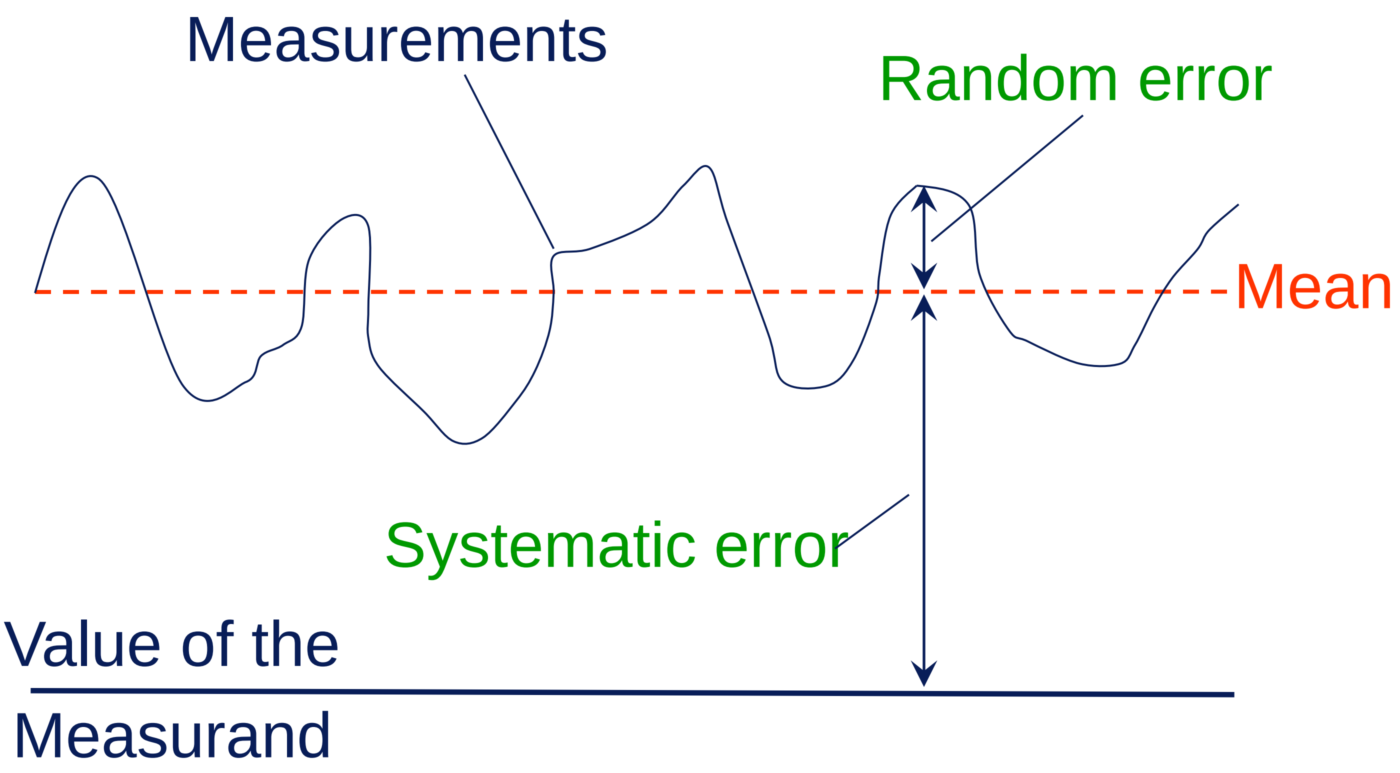
$$x_{i,k} - x_i$$

Systematic error

Mean that would result from an infinite number of measurements of the same measurand carried out under repeatability conditions
minus
the value of the measurand.

$$x_i - x$$

Error of Measurement - Illustration



Recognize systematic error – still unknown systematic error

Uncertainty Evaluation Procedure – Summary

1. Express functional relationship between the measurand **(y)** and input parameters **(x_i) —** your measurement equation**.**

   $$y = f(x_1, x_2, \ldots, x_i, \ldots, x_m)$$

2. Determine values of input parameters x_i [Statistical analysis or other means].

3. Evaluate standard uncertainty $u(x_i)$ of each input x_i (Type A and/or Type B technique).

4. Calculate the value of measurand **(y)** from the functional relationship

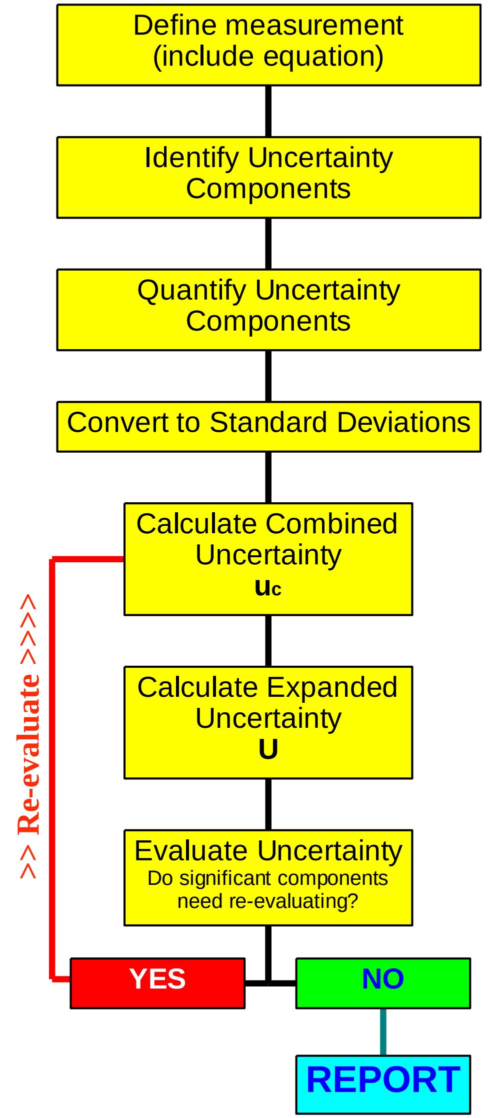
5. Determine the combined standard uncertainty $u_c(y)$ from the standard uncertainties associated with each input parameter **(x_i)**.

6. Calculate the expanded standard uncertainty **(U)** as the combined standard uncertainty $u_c(y)$ times the coverage factor **(k).**

7. Report the value of the measurand y **and** the combined standard uncertainty $u_c(y)$ **or the** expanded uncertainty U **.**

Uncertainty Analysis Procedure

1. Specify measurement equation
2. Identify uncertainty components
3. Quantify uncertainty sources
4. Convert to standard deviations
5. Combined uncertainty
6. Expanded uncertainty
7. Evaluate tolerance requirements
8. Report uncertainty



EURACHEM Flow Chart (modified)

1. Specify Measurement Equation

Express functional relationship between the measurand (*y*) and input parameters (x_i).

$$y = f(x_1, x_2, \ldots, x_i, \ldots, x_m)$$

Examples

Mass

$$M_x = M_s + d + \rho_a \cdot (V_x - V_x) + drift/bias + unrelated\ corrections$$

Signal

$$S_c \cong \frac{A_1 \cdot \cos\theta_1 \cdot A_2 \cdot \cos\theta_2}{D^2} \cdot G \cdot R(\lambda) \cdot \tau(\lambda) \cdot L_b(\lambda, T) \cdot \Delta\lambda$$

Temperature

$$T_{BB} = \frac{c_2}{n_\lambda \cdot \lambda \cdot \ln\left[1 + \frac{\varepsilon_{\lambda,RT} \cdot c_{1L}}{n_\lambda^2 \cdot \lambda^5 \cdot L_{WS4} \cdot r_4} \cdot \frac{(C_A \cdot C_L \cdot C_S \cdot G)_{WS}}{(C_A \cdot C_L \cdot C_S \cdot G)_{BB}}\right]}$$

2. Identify Uncertainty Sources

Determine values of input parameters $\boldsymbol{x_i}$
[Statistical analysis or other means].

$$S_c \cong \frac{A_1 \cdot \cos\theta_1 \cdot A_2 \cdot \cos\theta_2}{D^2} \cdot G \cdot R(\lambda) \cdot \tau(\lambda) \cdot L_b(\lambda, T) \cdot \Delta\lambda$$

Component Symbol	Description of Input
A_1	Blackbody area
θ_1	Blackbody angle
A_2	Detector area
θ_2	Detector angle
D	Distance between blackbody and detector
G	Amplifier gain

Component Symbol	Description of Input
R	Detector Spectral Response
τ	Filter transmissivity
T	Temperature
λ	Wavelength
$\Delta\lambda$	Wavelength bandwidth

Additional components or correction factors??

3. Quantify uncertainty estimates

Component	Uncertainty Estimate	Units	Reference	Type	Distribution
A_1	$1.96\ \ 10^{-3}$	m^2	Caliper	A	Normal
θ_1	0	o	Alignment	B	Rectangular
A_2	$3.323\ \ 10^{-5}$	m^2	Caliper	A	Normal
θ_2	0	o	Alignment	B	Rectangular
D	0.415	m	Ruler	B	Rectangular
G	0.01	V/A	NIST	A	Normal
R	0.0001	A/W	NIST	A	Normal
τ	0.001		NIST	A	Normal
T	0.005	K	Temperature	A	Normal
λ	$5\ \ 10^{-9}$	m	Filter	A	Normal
$\Delta\lambda$	$5\ \ 10^{-9}$	m	Filter	A	Normal

$$S_c \cong \frac{A_1 \cdot \cos\theta_1 \cdot A_2 \cdot \cos\theta_2}{D^2} \cdot G \cdot R(\lambda) \cdot \tau(\lambda) \cdot L_b(\lambda, T) \cdot \Delta\lambda$$

Classification of Uncertainty Components

Due to random effects (Type A)	Due to systematic effects (Type B)
Give rise to possible random error in the unpredictable result of the current measurement process. Usually decrease with increasing number of observations	Give rise to possible systematic error in the result due to recognized effects in the current measurement process.

Correction and Correction Factor

Used to account for systematic error

Correction

Value added algebraically to the uncorrected result of a measurement to compensate for systematic error.

Correction = - (systematic error)

Correction Factor

Numerical factor by which the uncorrected result of a measurement is multiplied to compensate for systematic error.

E.g. Linearity, offset, shunt resistance, drift, stray light

4. Convert to **S**tandard Uncertainties

Evaluate standard uncertainty $\boldsymbol{u(x_i)}$ of each input $\boldsymbol{x_i}$
(Type A or Type B technique).

Component	Sensitivity Coefficient	Standard Uncertainty	Units
A_1	S_c/A_1	4 10^{-5}	V
θ_1	$-\theta_1{}^2 \cdot (S_c/\theta)$	0	V
A_2	S_c/A_2	1 10^{-6}	V
θ_2	$-\theta_2{}^2 \cdot (S_c/\theta)$	0	V
D	$-2 \cdot (S_c/D)$	0.0012	V
G	S_c/G	0	V
R	S_c/R	0.0007	V
τ	S_c/τ	0.005	V
T	$[c_2/(\lambda T)] \cdot (S_c/T)$	2	V
λ	$\{[c_2/(\lambda T)]-5\} \cdot (S_c/\lambda)$	0.001	V
$\Delta\lambda$	$S_c/\Delta\lambda$	0.0001	V

Standard Uncertainty

Measurand (y) determined from m input parameters x_i through functional relationship $f(x_1, x_2, \ldots, x_i, \ldots, x_m)$

Example: Radiometer signal measurement

$$v \cong \Gamma \cdot G \cdot s(\lambda) \cdot \tau(\lambda) \cdot L(\lambda, T) \cdot \Delta\lambda$$

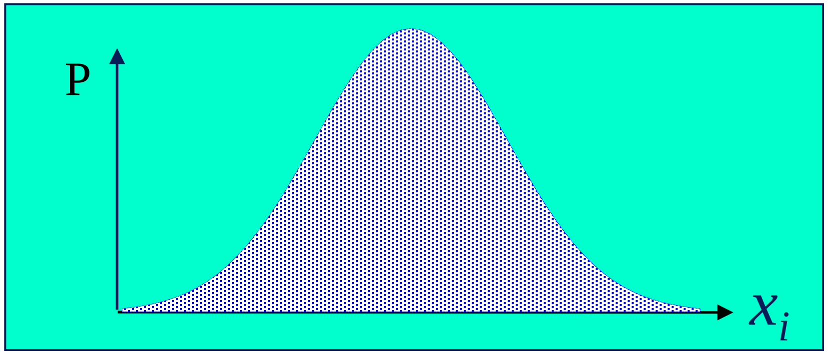
Input parameters are throughput (Γ), gain (G), responsivity (**s**), transmittance (τ), radiance (**L**), wavelength (λ), bandwidth ($\Delta\lambda$) and source temperature (**T**)

Standard uncertainty

Estimated standard deviation associated with each input estimate x_i, denoted $u(x_i)$

Example: $u(\mathbf{G})$, $u(\mathbf{l})$, $u(T)$, etc.

Standard uncertainty $u(x_i)$ determined from probability distribution (P) of parameter (x_i)



Statistical Parameter – Sample Mean

Mean

$$\bar{x}_i = \frac{1}{n}\sum_{k=1}^{n} x_{i,k}$$

Sum of all the sample values ($x_{i,k}$) divided by the size of the sample (n)

Example

Five voltage readings: 0.9, 1.2, 1.1, 0.8, 1.0

Size of the sample = 5

Sample mean = (0.9 + 1.2 + 1.1 + 0.8 + 1.0)/5 = **1.0 [V]** .

Statistical Parameter – Sample Variance

Variance: $$\sigma^2(x_{i,k}) = \frac{1}{n-1}\sum_{k=1}^{n}(x_{i,k} - x_i)^2$$

Sum of the squares of the deviations of the sample values ($x_{i,k}$) from the mean value (x_i), divided by (n - 1).

Measures the spread or dispersion of the sample values, and is positive.

Variance of the mean $$\sigma^2(x_i) = \frac{\sigma^2(x_{i,k})}{n}$$

Example: Five voltage readings: 0.9, 1.2, 1.1, 0.8, 1.0; Sample mean = 1.0 [V]

Variance = [(0.9-1.0)^2 + (1.2-1.0)^2 + (1.1-1.0)^2 + (0.8-1.0)^2 + (1.0-1.0)^2]/(5-1) = **0.025 [V^2]**

Variance of the mean = 0.025/5 = **0.005 [V^2]**

Type A Evaluation of Standard Uncertainty

Standard deviation = (Variance)$^{1/2}$ = $\sigma(x_{i,k})$
(Positive square root of the sample variance)

Standard deviation of the mean: $\sigma(x_i) = \sigma(x_{i,k}) / n^{1/2}$

Standard uncertainty $u(x_i)$ = $\sigma(x_i)$
(Standard deviation divided by the square root of the number of samples)
Relative standard uncertainty = $u(x_i)/x_i$

Example: Five voltage readings: 0.9, 1.2, 1.1, 0.8, 1.0
Sample mean = 1.0 [V], Variance = 0.025 [V^2], Variance of the mean = 0.005 [V^2]

Standard deviation = (Variance)$^{1/2}$ = $0.025^{1/2}$ = **0.158 [V]**
Standard uncertainty = Standard deviation of the mean = $0.158/5^{1/2}$ = **0.071 [V]**
Relative standard uncertainty = 0.071/1.0 = **0.071**

Type B Evaluation of Standard Uncertainty

Evaluated based on scientific judgment, experience, manufacturer's specification, data from other sources (reports, handbooks)

Examples

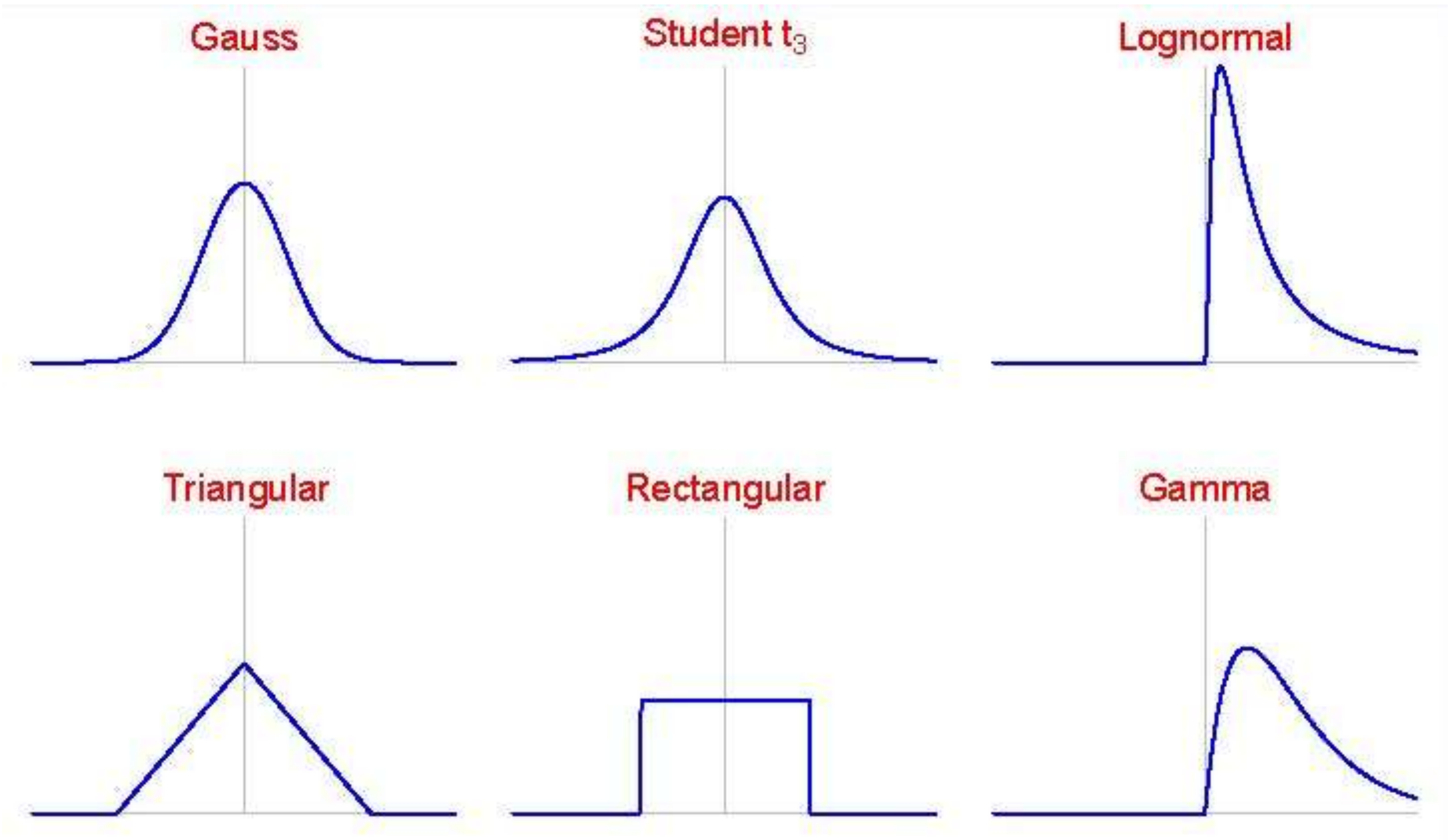
Convert a quoted uncertainty (with a stated multiple) to a standard uncertainty by dividing by the multiple

Convert a quoted uncertainty (with a specified confidence level, such as 95 % or 99 %) to a standard uncertainty by dividing by the appropriate factor for a normal distribution

Computational methods

Model the quantity by an assumed probability distribution such as normal, rectangular or triangular.

Univariate Probability Distributions



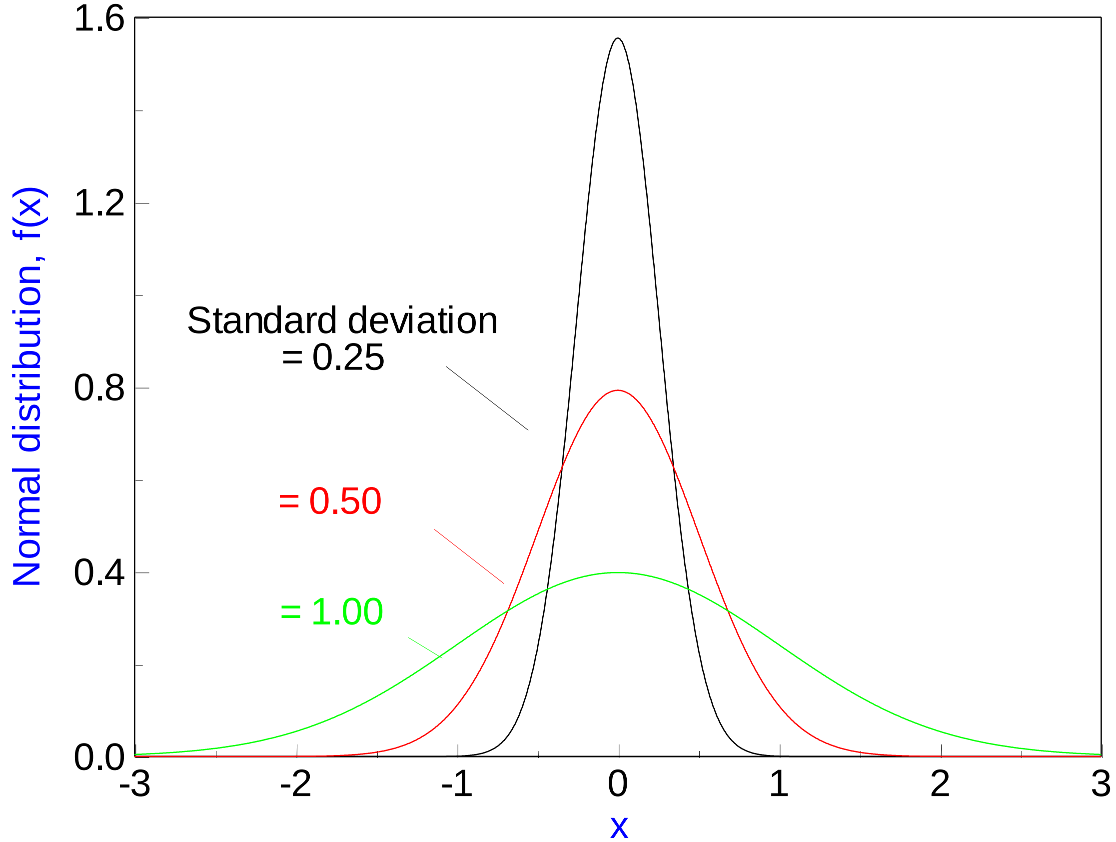
Normal Probability Distribution

Most widely used probability distribution function in many applications and most useful asymptotic approximation for more complicated distributions

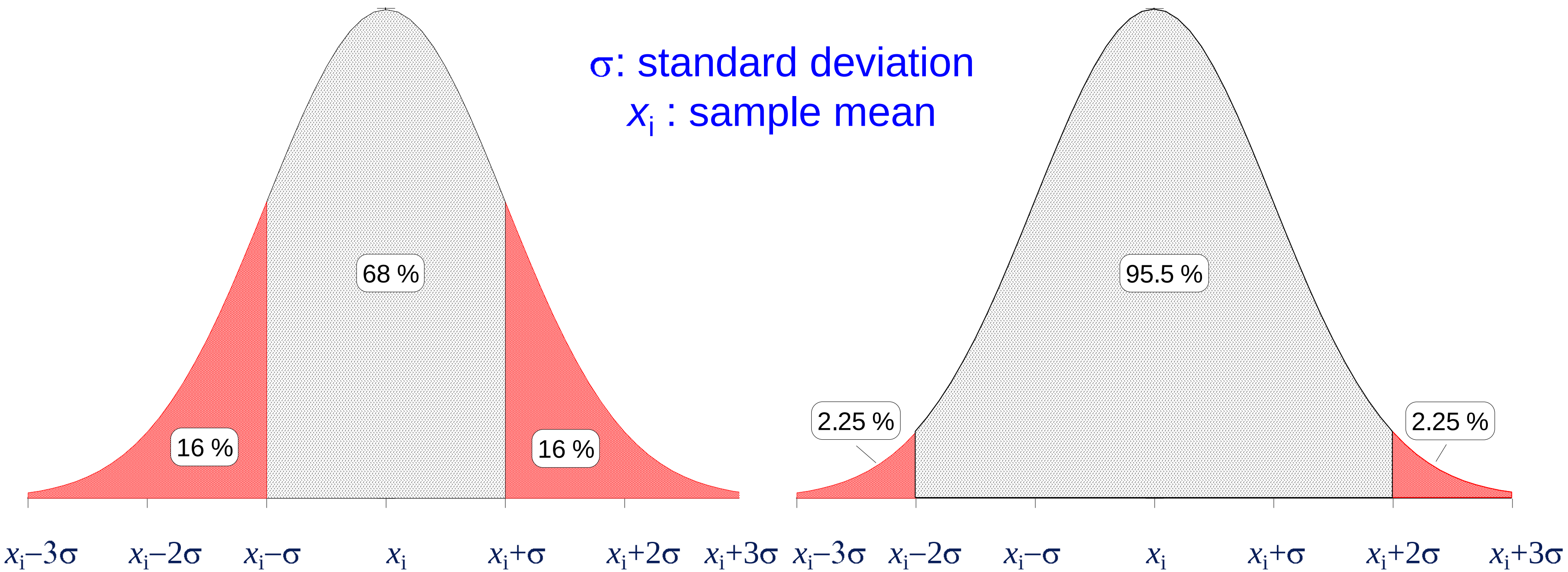
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

σ : standard deviation,
μ : sample mean

Smaller standard deviation → narrower distribution.
Larger standard deviation → wider distribution.



Normal Probability Distribution



Probability that x lies between $(x_i - \sigma)$ and $(x_i + \sigma)$ is 68 %.

For large number of observations, about 68 % of the values lie in this range, **OR**

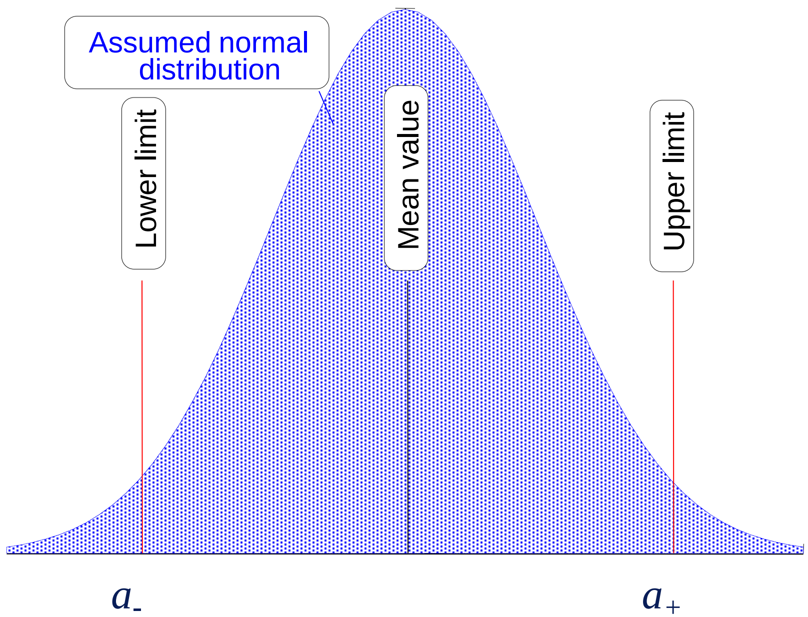
a value deviating more than σ from mean x_i will occur about once in 3 trials.

Probability that x lies between $(x_i - 2\sigma)$ and $(x_i + 2\sigma)$ is 95.5 %.

For large number of observations, about 95 % of the values lie in this range, **OR**

a value deviating more than 2σ from mean x_i will occur about once in 20 trials.

Type B Calculation – Normal Distribution



Center of the limits
$= (a_+ + a_-)/2$

Half width of interval
$a = (a_+ - a_-)/2$

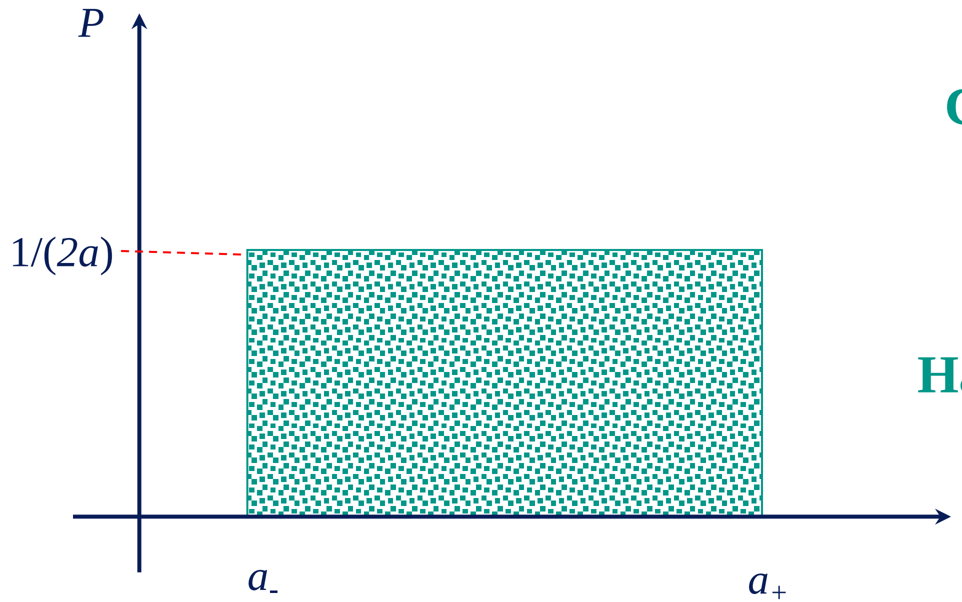
Estimated the lower limit (a_-), and the upper limit (a_+) of the quantity.
Best estimated value of the quantity (mean) = center of the limits

50.0 % probability, value lies in the interval a_- to a_+, **then $u(x_i) = 1.48\, a$**
67.7 % probability, value lies in the interval a_- to a_+, **then $u(x_i) = a$**
99.7 % probability, value lies in the interval a_- to a_+, **then $u(x_i) = a/3$**

Type B Calculation – Rectangular Distribution

Equal probability the value lies in the interval a_- and a_+ is 100 % and zero outside

(Reasonable default model in the absence of any other information)



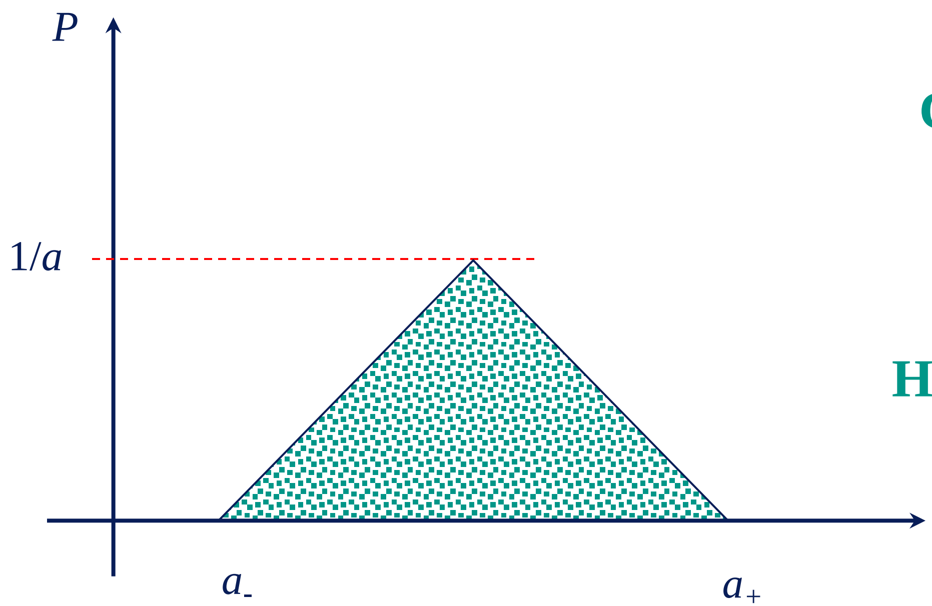
Center of the limits
$= (a_+ + a_-)/2$

Half width of interval
$a = (a_+ - a_-)/2$

Best estimated value of the quantity (mean) = center of the limits with **$u(x_i) = a/3^{½}$ or $[\text{max-min}]/(12)^{1/2}$**

Type B Calculation – Triangular Distribution

Probability the value lies in the interval a_- and a_+ is 100 % and zero outside



Center of the limits
$= (a_+ + a_-)/2$

Half width of interval
$a = (a_+ - a_-)/2$

Best estimated value of the quantity (mean) = center of the limits
with
$u(x_i) = a/6^{½}$ or $[\text{max-min}]/(24)^{1/2}$

5. Calculate u_c

Determine combined standard uncertainty $u_c(y)$ from standard uncertainties associated with each input parameter (x_i).

Component	Value	Standard Uncertainty	Relative Uncertainty
A_1	$1.96\ \ 10^{-3}$	$4\ \ 10^{-5}$	2.0 %
A_2	$3.32\ \ 10^{-5}$	$1\ \ 10^{-6}$	3.0 %
D	0.415	0.0012	0.6 %
R	0.3467	0.0007	0.2 %
τ	0.59	0.005	0.8 %
T	1241	2	3.6 %
λ	0.5137	0.001	3.4 %
$\Delta\lambda$	0.0088	0.0001	1.1 %
		Standard Uncertainty	6.3 %

Calculating combined standard uncertainty

Functional relationship between measurand and input parameters

$y = f(x_1, x_2, \ldots, x_i, \ldots, x_m)$

Combined standard uncertainty, $u_c(y)$

Represents the estimated standard uncertainty of the measurand y.

given by

Law of Propagation of Uncertainty

$$u_c^2(y) = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

$\partial f / \partial x_i$: sensitivity coefficient, $u(x_i)$: standard uncertainty of x_i

$u(x_i)/x_i$: relative standard uncertainty of x_i

$u(x_i, x_j)$: covariance of x_i and x_j $= u(x_i) \cdot u(x_j) \cdot r(x_i, x_j)$

$r(x_i, x_j)$: correlation coefficient $r = 0$, if uncorrelated $[-1 \leq r \leq 1]$

Additive and multiplicative functions

Additive function (Two independent random variables x_1 and x_2)
Use standard uncertainties to calculate combined standard uncertainty

$$y = a \cdot x_1 + b \cdot x_2$$

$$\frac{\partial y}{\partial x_1} = a \qquad \frac{\partial y}{\partial x_2} = b$$

$$u_c^2(y) = a^2 \cdot u^2(x_1) + b^2 \cdot u^2(x_2)$$

$$y = a \cdot x_1 \cdot x_2$$

$$\frac{\partial y}{\partial x_1} = a \cdot x_2 \qquad \frac{\partial y}{\partial x_2} = a \cdot x_1$$

$$u_c^2(y) = a^2 \cdot x_2^2 \cdot u^2(x_1) + a^2 \cdot x_1^2 \cdot u^2(x_2)$$

$$\frac{u_c^2(y)}{y^2} = \frac{u^2(x_1)}{x_1^2} + \frac{u^2(x_2)}{x_2^2}$$

Multiplicative function (Two independent random variables x_1 and x_2)
Use relative standard uncertainties to calculate combined standard uncertainty

Uncorrelated vs. Correlated

$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} r(x_i, x_j)\, u(x_i)\, u(x_j)$$

Example: $y = a_1 x_1 + a_2 x_2$

Sensitivity Coefficients: $\partial y/\partial x_1 = a_1$, $\partial y/\partial x_2 = a_2$

Uncorrelated, $r(x_i, x_j) = 0$

$$u_c^2(y) = a_1^2 u^2(x_1) + a_2^2 u^2(x_2)$$ sum-of-squares

Correlated, $r(x_i, x_j) = \pm 1$

$$u_c^2(y) = a_1^2 u^2(x_1) + a_2^2 u^2(x_2) \pm 2 a_1 a_2 u(x_1) u(x_2)$$

$$= \left[a_1 u(x_1) \pm a_2 u(x_2) \right]^2$$ square-of-sum or -difference

Correlations

Example: $y = a\frac{x_1}{x_2}$ Sensitivity Coefficients: $\partial y/\partial x_1 = a/x_2$, $\partial y/\partial x_2 = -ax_1/x_2^2$

Sensitivity Coefficients: $\partial y/\partial x_1 = a/x_2$, $\partial y/\partial x_2 = -ax_1/x_2^2$

Uncorrelated: $r(x_i, x_j) = 0$ sum-of-squares

$$\frac{u_c^2(y)}{y^2} = \frac{u^2(x_1)}{x_1^2} + \frac{u^2(x_2)}{x_2^2}$$

Correlated: $r(x_i, x_j) = 1$ square-of-difference

$$\frac{u_c^2(y)}{y^2} = \frac{u^2(x_1)}{x_1^2} + \frac{u^2(x_2)}{x_2^2} - 2\frac{u(x_1)}{x_1}\frac{u(x_2)}{x_2} = \left[\frac{u(x_1)}{x_1} - \frac{u(x_2)}{x_2}\right]^2$$

For correlated, if $u(x_1) = u(x_2)$ and $x_1 = x_2$ then $u_c(y) = 0$

Example – Correlations with Standard

Can the uncertainty from the standard be reduced by using additional standards?

$u(\Phi_s) = 2$ W, if use four expect $u(\Phi_{s,avg}) = 2\text{ W}/\sqrt{4} = 1$ W

since $\Phi_{s,avg} = ¼(\Phi_{s,1} + \Phi_{s,2} + \Phi_{s,3} + \Phi_{s,4})$

Not correct, because of correlations between uncertainties caused by systematic effects, which dominate for standards

Recall that combined uncertainty for correlation coefficient $r = 1$ is the square-of-sums, so

$$u_c^2(\Phi_{s,avg}) = \left(\frac{1}{4}\right)^2 \left[u(\Phi_{s,1}) + u(\Phi_{s,2}) + u(\Phi_{s,3}) + u(\Phi_{s,4})\right]^2$$

$$u_c(\Phi_{s,avg}) = 2\text{ W}$$

6. Calculate *U*

Calculate expanded standard uncertainty (U) as combined standard uncertainty $u_c(y)$ times coverage factor (k).

Component	Value	Standard Uncertainty	Relative Uncertainty
A_1	$1.96\ \ 10^{-3}$	$4\ \ 10^{-5}$	2.0 %
A_2	$3.32\ \ 10^{-5}$	$1\ \ 10^{-6}$	3.0 %
D	0.415	0.0012	0.6 %
R	0.3467	0.0007	0.2 %
t	0.59	0.005	0.8 %
T	1241	2	3.6 %
λ	0.5137	0.001	3.4 %
$\Delta\lambda$	0.0088	0.0001	1.1 %
		Standard Uncertainty	6.3 %
		Coverage Factor, *k* = 2	2
		Expanded Uncertainty	12.6 %

Degrees of Freedom

1. Obtain y and $u_c(y)$
2. Estimate v_{eff} – effective degrees of freedom

$$v_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{m} \frac{c_i^4 u^4(x_i)}{v_i}} \qquad c_i = \partial f / \partial x_i$$

$v = n – 1$ for random effects, ∞ for systematic effects

(Welch-Satterthwaite formula)

3. Obtain the t-factor for required level of confidence p from a table of values of $t_p(v)$ from the t-distribution
4. Take $k_p = t_p(v_{eff})$ and calculate $U_p = k_p u_c(y)$

Table of $t_p(v)$

	Fraction p in percent					
v	68.27	90	95	95.45	99	99.73
1	1.84	6.31	12.71	13.97	63.66	235.80
10	1.05	1.81	2.23	2.28	3.17	3.96
100	1.005	1.660	1.984	2.025	2.626	3.077
∞	1.000	1.645	1.960	2.000	2.576	3.000

Expanded Uncertainty

Measure of uncertainty defining an *interval* about the result y within which the measurand is confidently believed to lie.

Expanded uncertainty (U) = Coverage factor (k) **x** Combined uncertainty $u_c(y)$

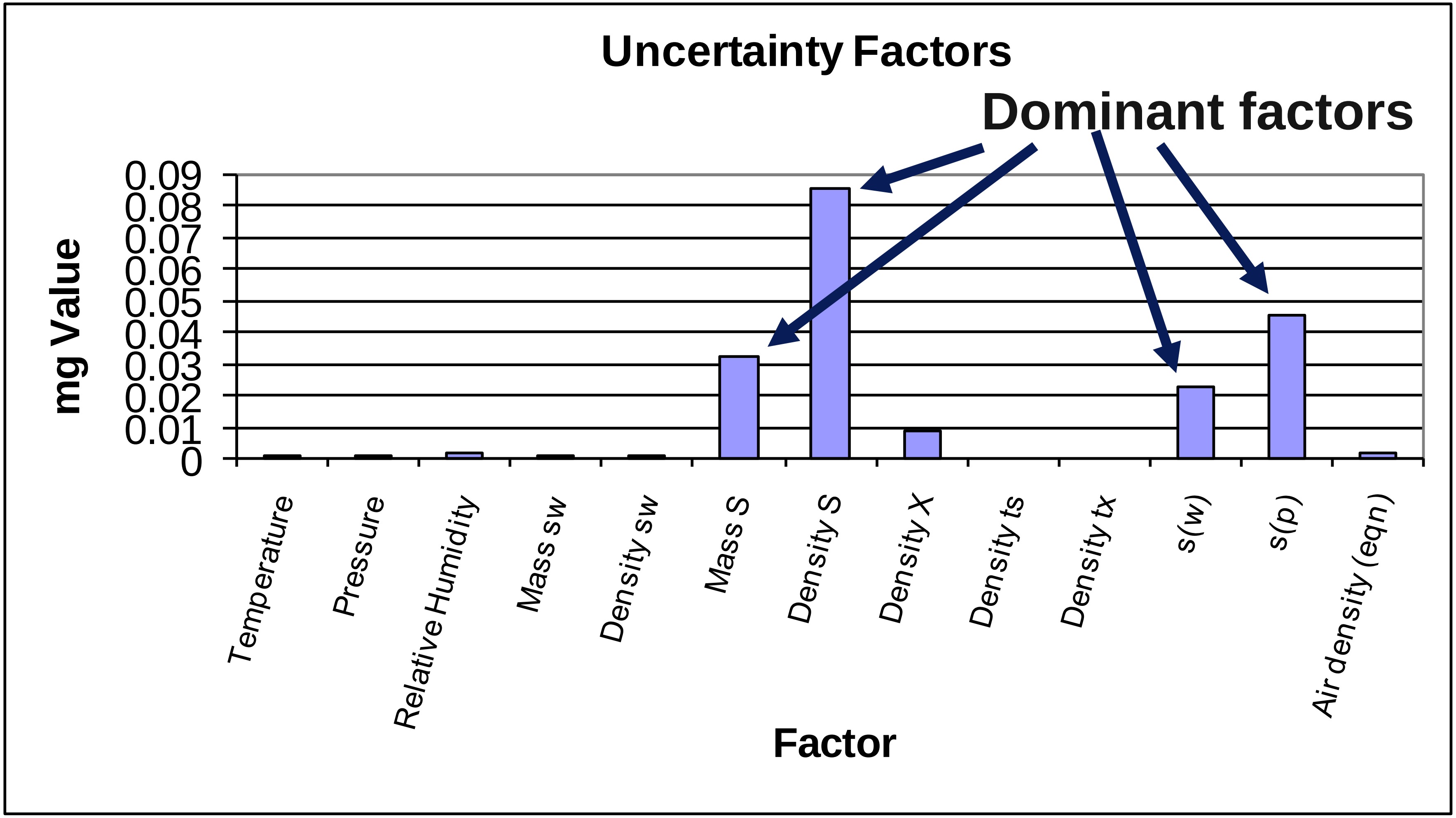
Coverage factor k	Confidence level for a normal probability distribution
1.000	68.27 %
1.645	90.00 %
1.960	95.00 %
2.000	**95.45 %**
2.576	99.00 %
3.000	99.73 %

1 out of 20 times should fall outside uncertainty budget

7. Evaluate Expanded Uncertainty

- Is it reasonable? Does it make sense?
- Have all components been included? If not, have statements to that effect been included? What are the assumptions, and how do they affect the uncertainties?
- Were the calculations done properly?
- Does it meet the customer's needs?
- Does it meet tolerance/specifications?
- Is it less than your "best measurement uncertainty"?
- Can it be validated with a proficiency test?
- Does the uncertainty need to be reduced to meet needs?
- Do one or two factors contribute significantly compared to the others?
- What factors can be reduced through alternative calibration sources, procedures, or equipment? How? How much will it cost?

Graphical Evaluation



8. Report Uncertainty

Report the value of the measurand *y* and specify the combined standard uncertainty $u_c(y)$ or the expanded uncertainty *U* with the associated coverage factor *k*.

- Assigned calibration value +/- associated uncertainty
- **Round to 2 significant digits**
- Must be completely defined – a Statement
  - Components
  - Type
  - Rationale
  - How combined
  - Level of confidence
  - List of items NOT included
  - Reference the GUM

Reporting the measurement result

A report of a measurement result should contain:

- the quantity, the value and the unit
- the associated <u>expanded</u> uncertainty (with k clearly stated)
    - the model of evaluation (measurement equation, if available)
    - the traceability and equivalence.

Example:

At the operational conditions ….. the luminous intensity of lamp No……

was measured traceable to NIST standards with a value **702,19 cd**

and an associated relative expanded uncertainty **0.34 %.**

The stated uncertainty was calculated according to the GUM by a

multiplication of the standard uncertainty with coverage factor ***k* = 2**.

Uncertainty Calculation Example

Signal measurement equation

$$S_x = \frac{V_x / V_{mx}}{V_s / V_{ms}} \cdot \frac{G_s}{G_x} \cdot S_s$$

S_x	Spectral responsivity of test detector
S_s	Spectral responsivity of standard detector
V	Voltage from test detector (x) or standard detector (s)
V_m	Voltage from monitor detector
G	Amplifier gain

Factors Contributing to Uncertainty

Examples of Sensitivity Coefficients

	Absolute	Relative
S_s :	$\dfrac{dS_x}{dS_s} = \left(\dfrac{V_x/V_{mx}}{V_s/V_{ms}} \cdot \dfrac{G_s}{G_x} \right)$	$\dfrac{dS_x}{S_x} = \dfrac{dS_s}{S_s}$
V_x :	$\dfrac{dS_x}{dV_x} = \left(\dfrac{1/V_{mx}}{V_s/V_{ms}} \cdot \dfrac{G_s}{G_x} \right) \cdot S_s$	$\dfrac{dS_x}{S_x} = \dfrac{dV_x}{V_x}$
V_s :	$\dfrac{dS_x}{dV_x} = -\left(\dfrac{V_x/V_{mx}}{V_s^2/V_{ms}} \cdot \dfrac{G_s}{G_x} \right) \cdot S_s$	$\dfrac{dS_x}{S_x} = \dfrac{-dV_s}{V_s}$
λ :	$\dfrac{dS_x}{d\lambda} = \dfrac{d(V_x/V_s)}{d\lambda} \cdot \dfrac{V_{ms}}{V_{mx}} \cdot \dfrac{G_s}{G_x} \cdot S_s$	$\dfrac{dS_x}{S_x} = \dfrac{d(V_x/V_s)}{d\lambda} \cdot \dfrac{d\lambda}{(V_x/V)_s}$

Combined Standard Uncertainty

Law of Propagation of Uncertainties

Using absolute uncertainties

$$u_c^2(S_x) = \left(\frac{dS_x}{dS_s}\right)^2 u^2(S_s) + \left(\frac{dS_x}{dV_x}\right)^2 u^2(V_x) + \left(\frac{dS_x}{dV_s}\right)^2 u^2(V_s) + \cdots + \left(\frac{dS_x}{d\lambda}\right)^2 u^2(\lambda)$$

Using relative uncertainties

$$\left(\frac{u_c(S_x)}{S_x}\right)^2 = \left(\frac{u(S_s)}{S_s}\right)^2 + \left(\frac{u(V_x)}{V_x}\right)^2 + \left(\frac{u(V_s)}{V_s}\right)^2 + \cdots + \left(\frac{d(V_x/V_s)}{d\lambda} \cdot \frac{u(\lambda)}{(V_x/V_s)}\right)^2$$

Uncertainty Summary

Quantity (Symbol)	Unit	Value	Probability Distribution		Standard uncertainty	Relative sensitivity	Relative uncertainty
Std. Resp. (S_s)	A/W	0.2848	Normal		0.0003	1/0.2848	0.11 %
Std. Signal (V_s)	V	2.000	Normal		0.002	1/2	0.10 %
Test Signal (V_x)	V	1.800	Normal		0.004	1/1.8	0.22 %
Monitor Signal (V_{ms})	V	1.100	Normal		0.001	1/1.1	0.09 %
Monitor Signal (V_{mx})	V	1.090	Normal		0.001	1/1.09	0.09 %
Std. Gain (G_s)	A/V	1.000×10^{-6}	Normal		10^{-10}	$1/10^{-6}$	0.01 %
Test Gain (G_x)	A/V	1.000×10^{-6}	Normal		10^{-10}	$1/10^{-6}$	0.01 %
Wavelength (λ)	nm	550.0	Rectangular		0.6	1/9	0.07 %
Test Resp. (S_x)	A/W	0.2587	Combined uncertainty in the responsivity				0.30 %
			Expanded uncertainty				0.60 %

Example – Dark Signal

$$V_x = 1.80 - 0.01 = 0.49$$

$$V_s = 2.00 - 0.01 = 0.99$$

$$u^2(V_x) = u^2(V_x) + u^2(V_x) = (0.007)^2 + (0.005)^2$$

$$u(V_x) = 0.009 \text{ V}$$

$$u(V_s) = 0.014 \text{ V}$$

Example – Signal Resolution

What if the signal had a resolution of only two decimal places, so that there was no variation in the signals?

V_s = 2.00 V, V_x = 1.80 V, V_{ms} = 1.10 V, V_{mx} = 1.09 V

What is the uncertainty in the net signal?

Assume a rectangular probability distribution with a = 0.005 V

Therefore, $u(V)$ = 0.005 / √3 = 0.003 V for all signals

$$\left(\frac{u_c(V)}{V}\right)^2 = \left(\frac{0.003}{2.00}\right)^2 + \left(\frac{0.003}{1.80}\right)^2 + \left(\frac{0.003}{1.10}\right)^2 + \left(\frac{0.003}{1.09}\right)^2$$

$$\frac{u_c(V)}{V} = 0.0045 = 0.45\%$$ up from 0.27 %

Useful EXCEL Functions

1. AVERAGE – average of values
2. COUNT – number of values
3. SQR – square root of value
4. STDEV – standard deviation of values
5. SUMSQ – sum of the squares of values
6. VAR – variance of values
7. Examples
   a) AVERAGE(0.9, 1.2, 1.1, 0.8, 1.0) = 1.0
   b) COUNT(0.9, 1.2, 1.1, 0.8, 1.0) = 5
   c) VAR(0.9, 1.2, 1.1, 0.8, 1.0) = 0.025
   d) STDEV(0.9, 1.2, 1.1, 0.8, 1.0) = 0.158
   e) STDEV/SQR(COUNT) = 0.071

Summary and Conclusions

1. Uncertainty analysis provides for a universal standardized approach for describing and evaluating uncertainties.

2. Type A and Type B evaluations are used to obtain standard uncertainties, which are in turn are used to obtain the combined uncertainty.

3. Uncertainty of a value must be stated.

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