Uncertainty of Integrated Quantities using Goniometric Data: What to do with the space between the measurements

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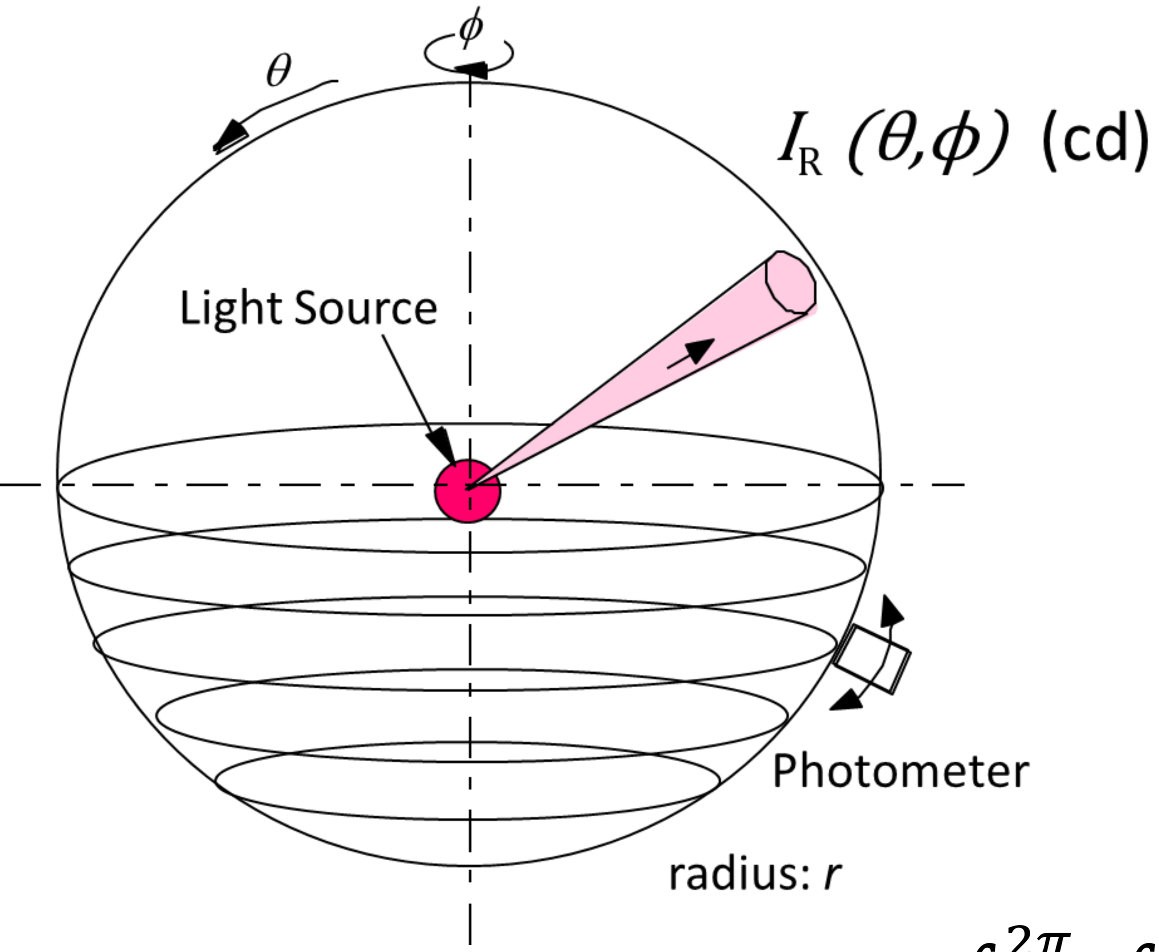
Current practice or methodology



$$\Phi_R = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_R(\theta,\phi)\, sin\,\theta d\theta\, d\phi$$

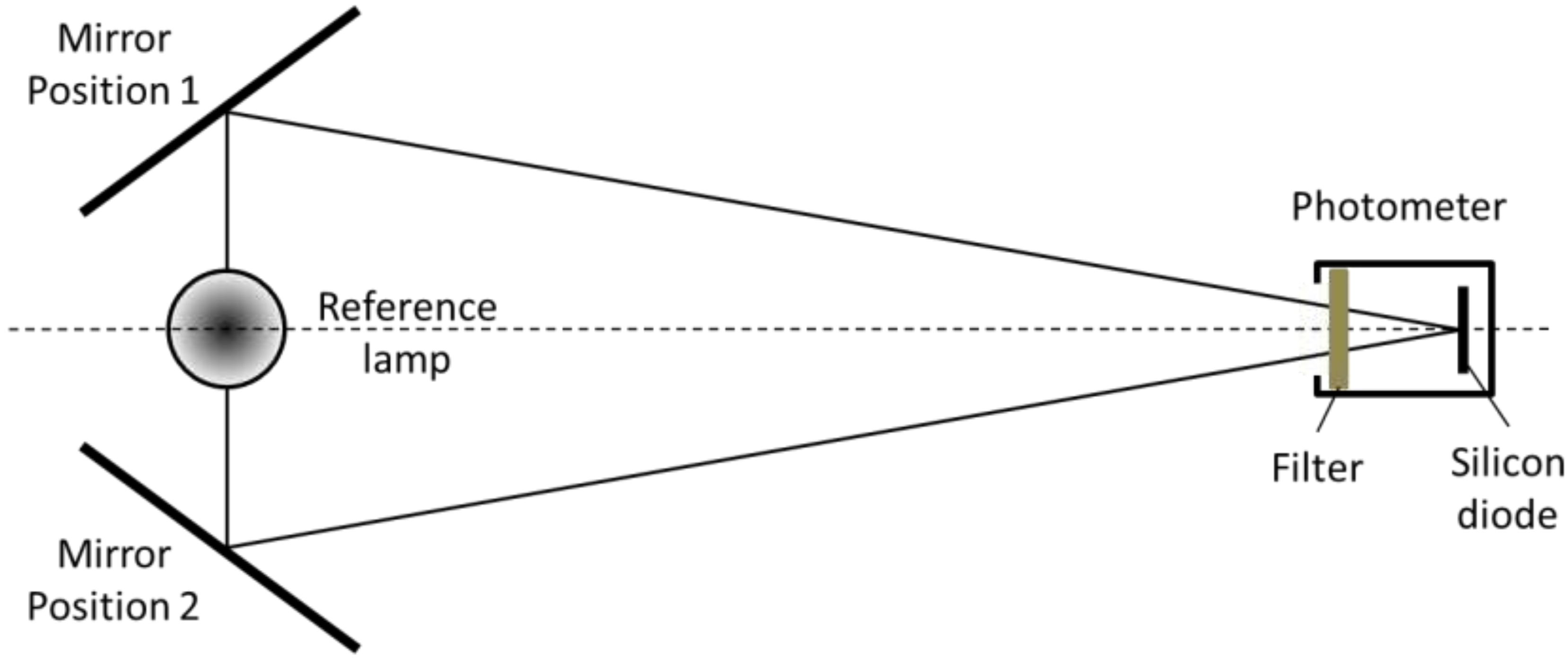
Measurement equation

$$\Phi_R = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_R(\theta,\phi)\, sin\,\theta d\theta\, d\phi$$

$$I_R(\theta,\phi) = \left(\left[y_R(\theta,\phi) - y_{R_d}(\theta,\phi\right] G(\theta,\phi) H(\theta)\right) / C_I$$

$$C_I = \frac{1}{\Phi_R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[y_R(\theta,\phi) - y_{R_d}(\theta,\phi)\right] G(\theta,\phi) H(\theta) \sin\theta d\theta\, d\phi$$

Relative angular photometer responsivity



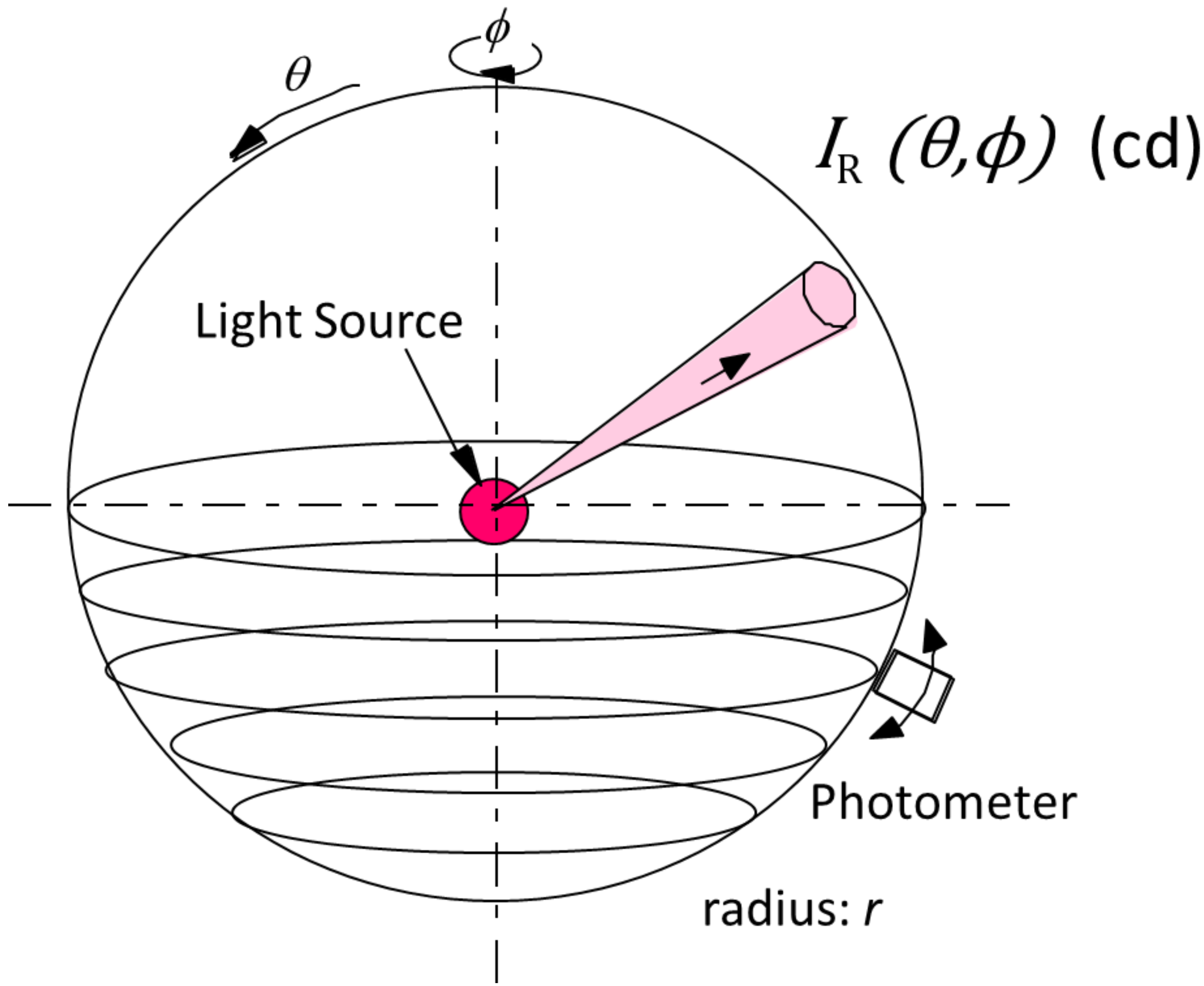
Integration to summation

$$y_I = 2\pi \sum_{0}^{\pi} \left(\left[\overline{y_R}(\theta)\bar{G}(\theta) - \overline{y_{R_d}}(\theta)\bar{G}(\theta) \right] \cdot H(\theta) \cdot \sin\theta \cdot \Delta\theta \right)$$

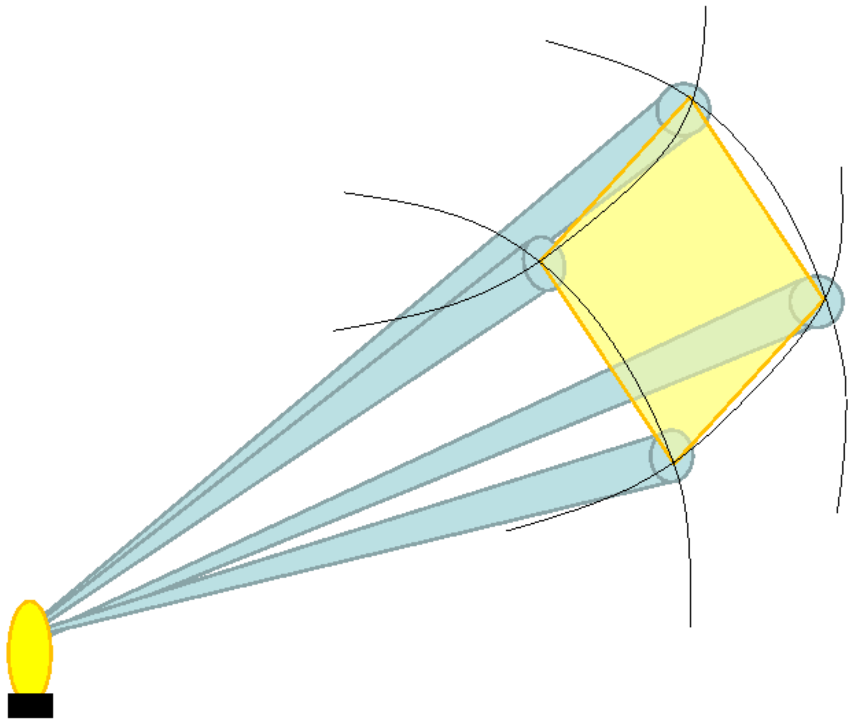
$$\overline{y_R}(\theta)\bar{G}(\theta) = \frac{1}{n} \sum_{\phi=0}^{2\pi} y_R(\theta,\phi) G(\theta,\phi)$$

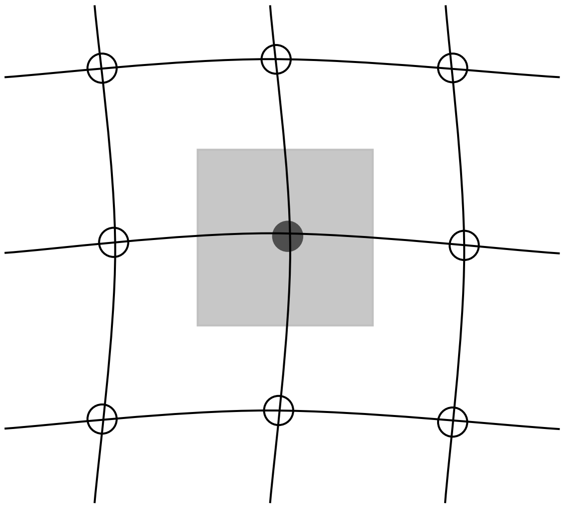
$$\overline{y_{R_d}}(\theta)\bar{G}(\theta) = \frac{1}{n} \sum_{\phi=0}^{2\pi} y_{R_d}(\theta,\phi) G(\theta,\phi)$$

So what's really happening?



What should we do?

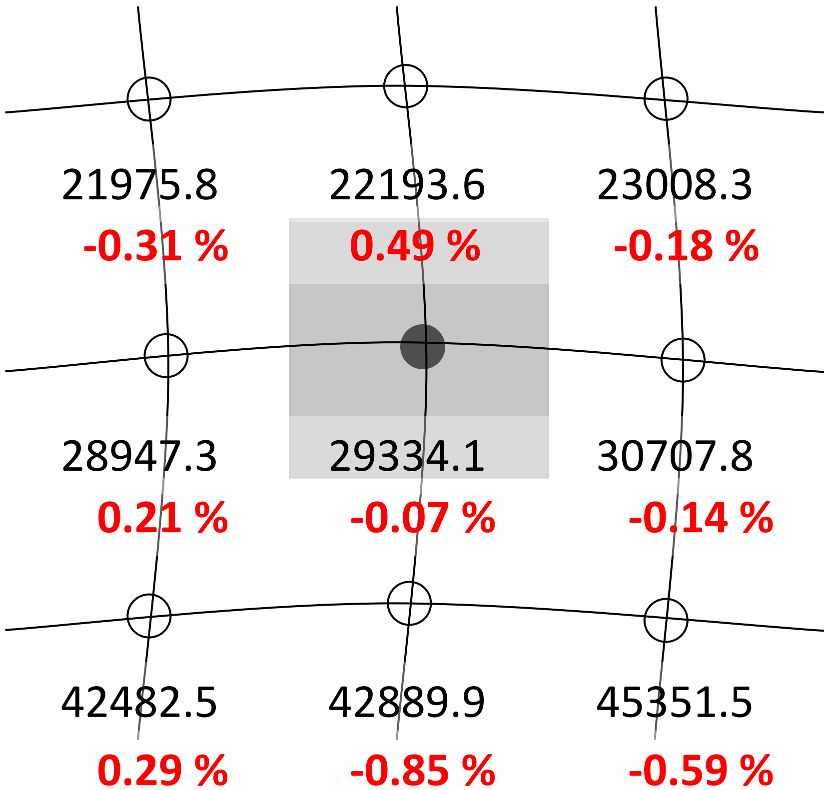




Model 2-D surface and calculate weighted solid angle and uncertainty of solid angle

Mathematical model

$$y(\theta,\phi) = k_0 + k_1\theta + k_2\phi + k_3\theta^2 + k_4\theta\phi + k_5\phi^2$$



- Perform least squares fit to data
- Weighted by relative uncertainty
- Results for 0.10 %

k_0 = 88,457 ± 375

k_1 = -1436.1 ± 405

k_2 = -668,854 ± 4031

k_3 = 3012.6 ± 156

k_4 = -25095 ± 1131

k_5 = 1,741,007 ± 13959

Fitting rigor - statistics

Null hypothesis – the model represents the distribution described by the measured data points

Chi-square test as a 'goodness of fit' test

Based on a confidence level ($\alpha < 0.05$)

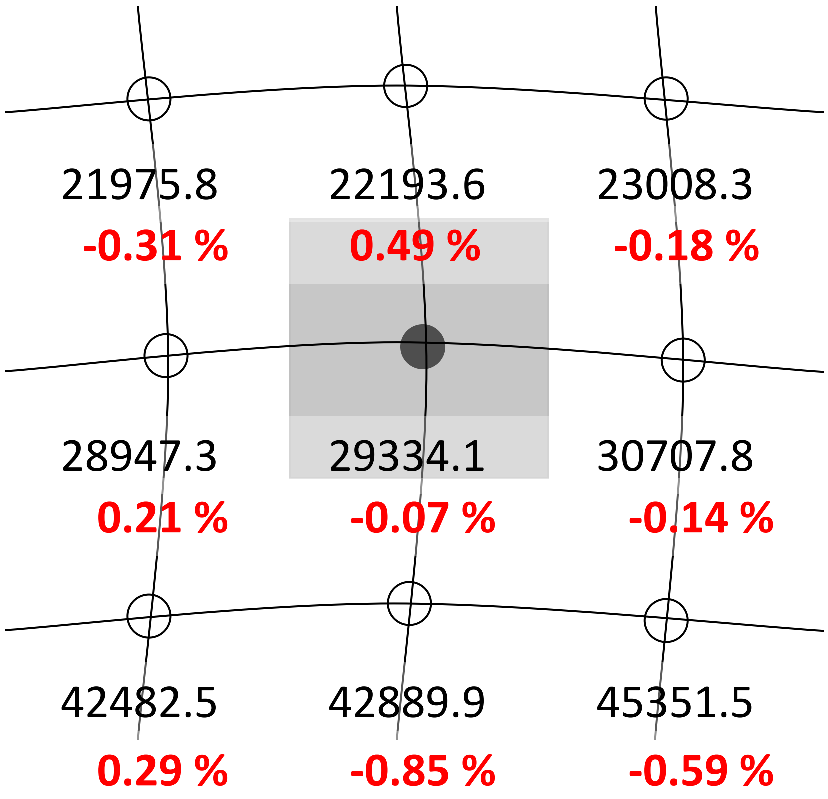
Degree of freedom ($\nu = 3$, 9 points – 6 parameters)

→ Critical value 7.815

Previous fit chi-square – 161.4, null hypothesis rejected soundly

Mathematical model

$$y(\theta, \phi) = k_0 + k_1\theta + k_2\phi + k_3\theta^2 + k_4\theta\phi + k_5\phi^2$$



Relative uncertainty = 0.46 %

k_0 = 88,457 ± 1727

k_1 = -1436.1 ± 1863

k_2 = -668,854 ± 18,543

k_3 = 3012.6 ± 720

k_4 = -25095 ± 5205

k_5 = 1,741,007 ± 64,215

Chi-square = 7.63

Weighted solid angle determination

Center point	191.3 lm
+ Four corner points	196.2 lm
10 x 10 weighted points	192.9 lm
Weighted average – integral	193.4 lm

$$\frac{\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} y(\theta,\phi) sin\theta d\theta d\phi}{\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} sin\theta d\theta d\phi} \cdot (\phi_2 - \phi_1) \cdot (sin\theta_2 - sin\theta_1)$$

Integral – just for completeness

$$\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} y(\theta,\phi) sin\theta d\theta d\phi$$

$$= k_0\phi cos\theta + k_1\phi(sin\theta - \theta cos\theta) - \frac{k_2}{2}\phi^2 cos\theta$$

$$+ k_3\phi(2cos\theta + 2\theta sin\theta - \theta^2 cos\theta) + \frac{k_4}{2}\phi^2(sin\theta - \theta cos\theta)$$

$$- \frac{k_5}{3}\phi^3 cos\theta$$

$$\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} sin\theta d\theta d\phi = (\phi_2 - \phi_1)\cdot(-cos\theta_2 + cos\theta_1)$$

Uncertainty of the solid angle

Confidence band – 95 % of the time the fit falls within a band

$$\pm t\left(n-p,1-\frac{\alpha}{2}\right)\sqrt{a^T C a} \qquad a=\left.\frac{\partial F}{\partial p}\right|x$$

Prediction band – 95 % of the measured points fall within a band

$$\pm t\left(n-p,1-\frac{\alpha}{2}\right)\sqrt{\chi^2+a^T C a}$$

Student's t distribution with *n-p* degrees of freedom having probability 1-α/2

a vector of partial derivatives of the model with respect to the coefficients evaluated at the given value of the independent variable

C covariance matrix

No analytical solution for a 2-D fit

Uncertainty of the solid angle

Monte Carlo analysis

Step 1 – Create a vector of random numbers from a normal distribution
Step 2 – Calculate a new set of parameters based on fit and standard deviation
Step 3 – Store the result of the solid angle determination
Step 4 – Repeat Step 1 – 3 for a while
Step 5 – Determine the mean result and the standard deviation of the results
Step 6 – Move to next solid angle

Questions

How many times to repeat?
What about correlation in the fitting parameters?

Correlation of fitting parameters

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

$$Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}}$$

Correlation Matrix	k_0	k_1	k_2	k_3	k_4	k_5
k_0	1.000	-0.537	-0.570	0.222	0.606	0.301
k_1	-0.537	1.000	0.117	-0.880	-0.402	-0.002
k_2	-0.570	0.117	1.000	0.000	-0.292	-0.928
k_3	0.222	-0.880	0.000	1.000	0.004	0.000
k_4	0.606	-0.402	-0.292	0.004	1.000	0.004
k_5	0.301	-0.002	-0.928	0.000	0.004	1.000

Cholesky decomposition

A well known fact from linear algebra is that any symmetric positive-definite matrix, M, may be written as

$$M = U^T D U$$

where U is an upper triangular matrix and D is a diagonal matrix with positive diagonal elements. Since our variance-covariance matrix, Σ, is symmetric positive-definite, we can therefore write

$$\Sigma = U^T D U$$

$$\Sigma = \left(U^T \sqrt{D}\right)\left(\sqrt{D} U\right)$$

$$\Sigma = \left(\sqrt{D} U\right)^T \left(\sqrt{D} U\right)$$

The matrix $C = \sqrt{D} U$ therefore satisfies $C^T C = \Sigma$. It is called the Cholesky decomposition of Σ.

Cholesky decomposition

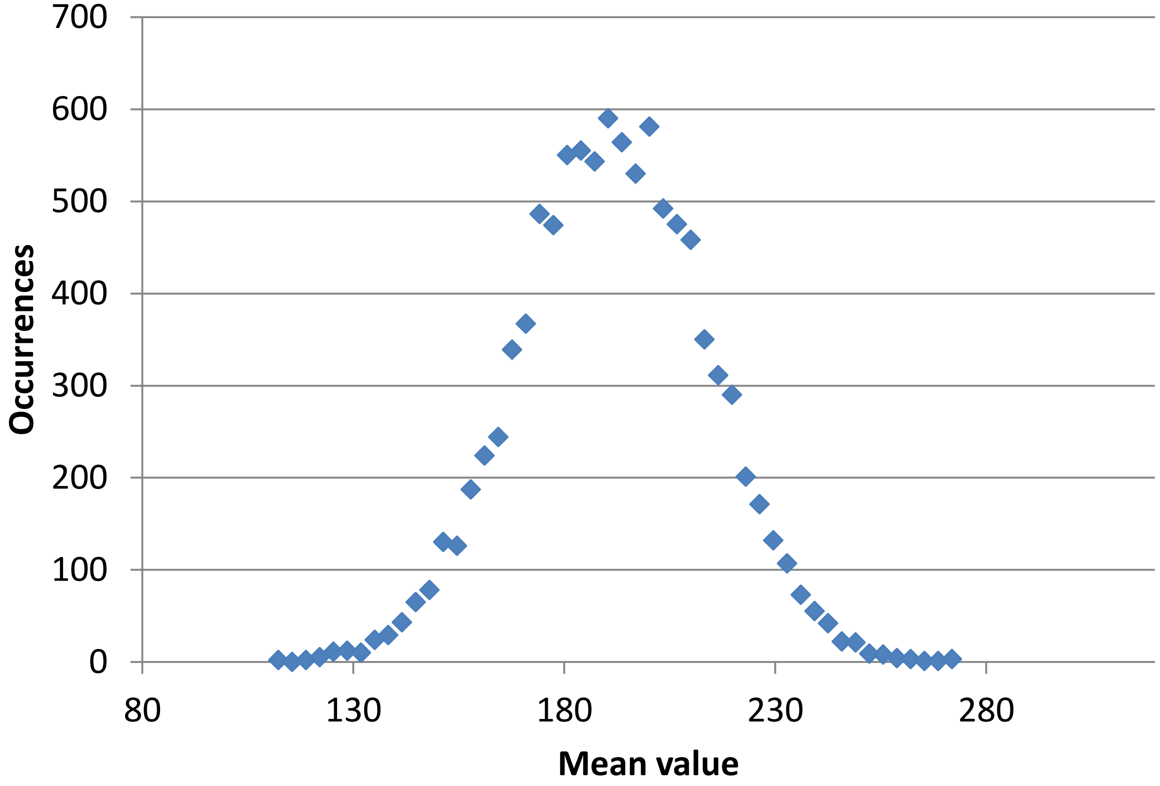
Cholesky	1	2	3	4	5	6
1	1.000	-0.537	-0.570	0.222	0.606	0.301
2	0.000	0.844	-0.225	-0.902	-0.091	0.189
3	0.000	0.000	0.790	-0.096	0.042	-0.904
4	0.000	0.000	0.000	0.357	-0.586	0.049
5	0.000	0.000	0.000	0.000	0.529	-0.178
6	0.000	0.000	0.000	0.000	0.000	0.151

$$\begin{bmatrix} 1.435 \\ 0.582 \\ -0.931 \\ -0.890 \\ -0.074 \\ 0.408 \end{bmatrix} = C^T \times \begin{bmatrix} 1.435 \\ 1.604 \\ 0.313 \\ 0.747 \\ -0.705 \\ -1.366 \end{bmatrix}$$

Number of Monte Carlo runs

Mean	Sdev
193.52	22.17
193.13	22.02
193.89	22.41
192.98	22.33
193.24	22.30
193.87	21.95
193.57	22.32
193.39	22.36
193.33	22.06
193.54	22.10
193.45	22.20
0.30	0.16
0.15 %	0.73 %

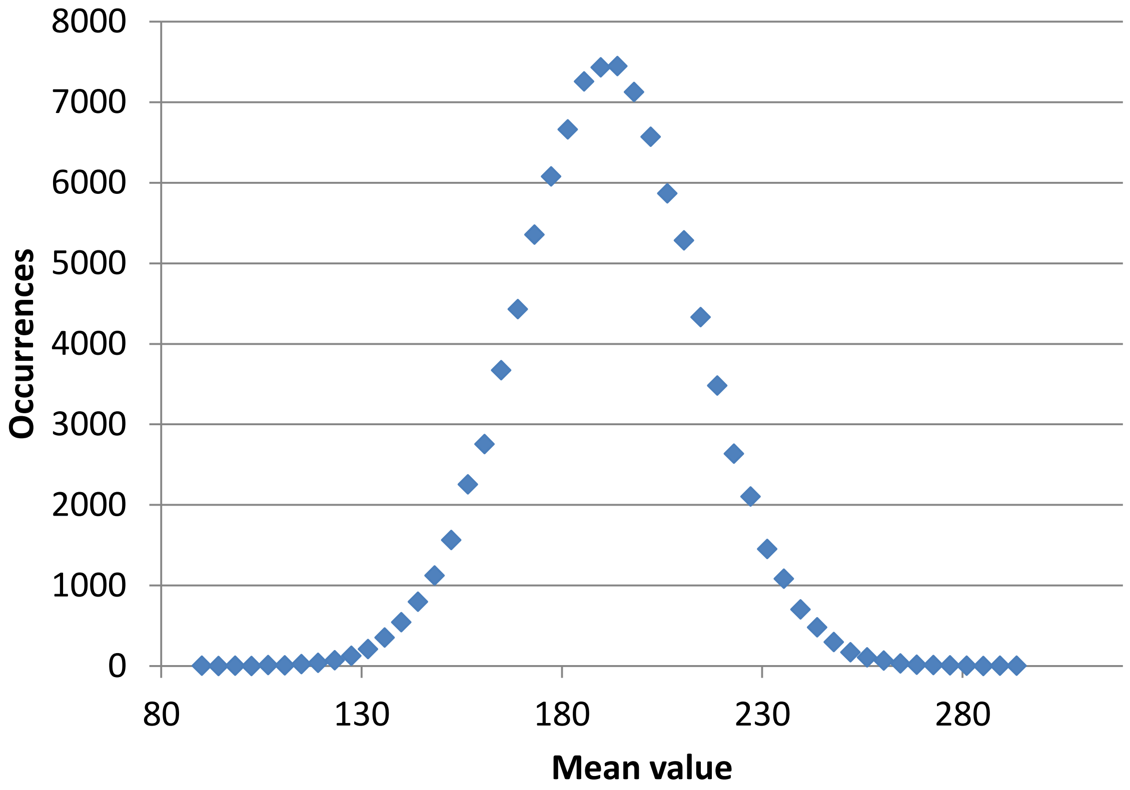
10000 runs



Number of Monte Carlo runs

Mean	Sdev
193.46	22.19
193.38	22.21
193.45	22.18
193.28	22.20
193.39	22.16
193.37	22.18
193.35	22.22
193.42	22.15
193.37	22.28
193.47	22.24
193.39	22.20
0.06	0.04
0.03 %	0.17 %

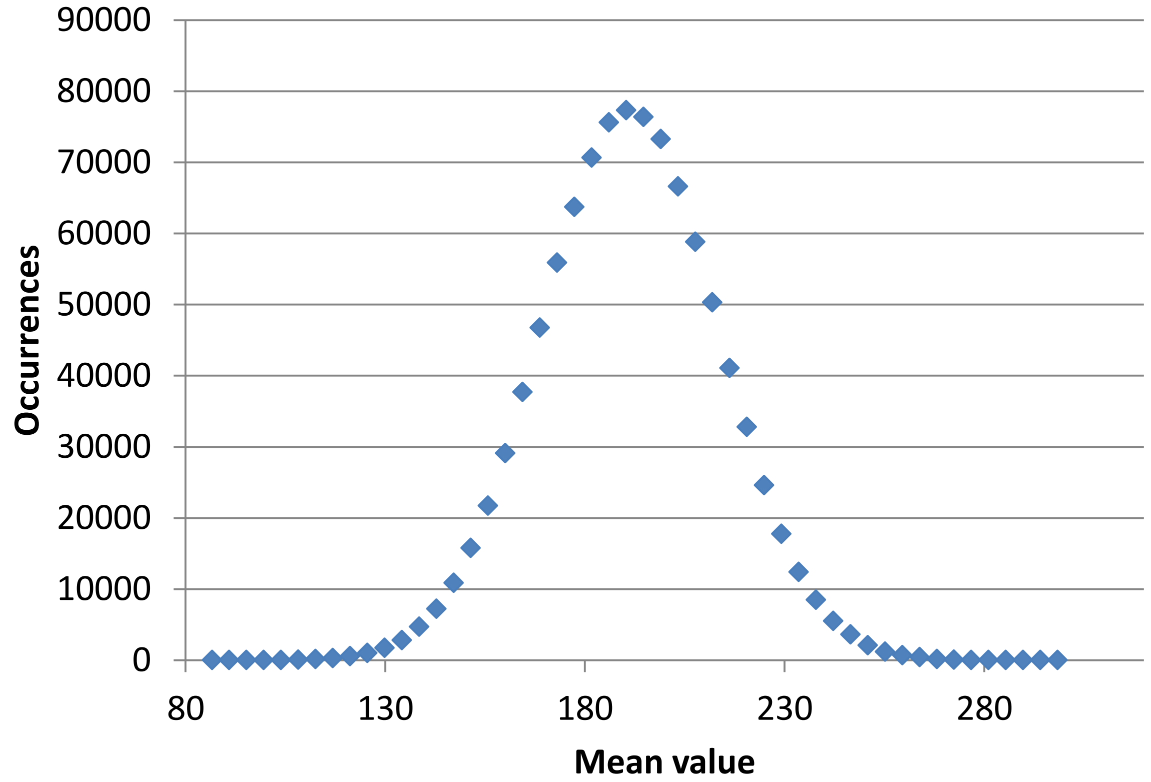
100000 runs



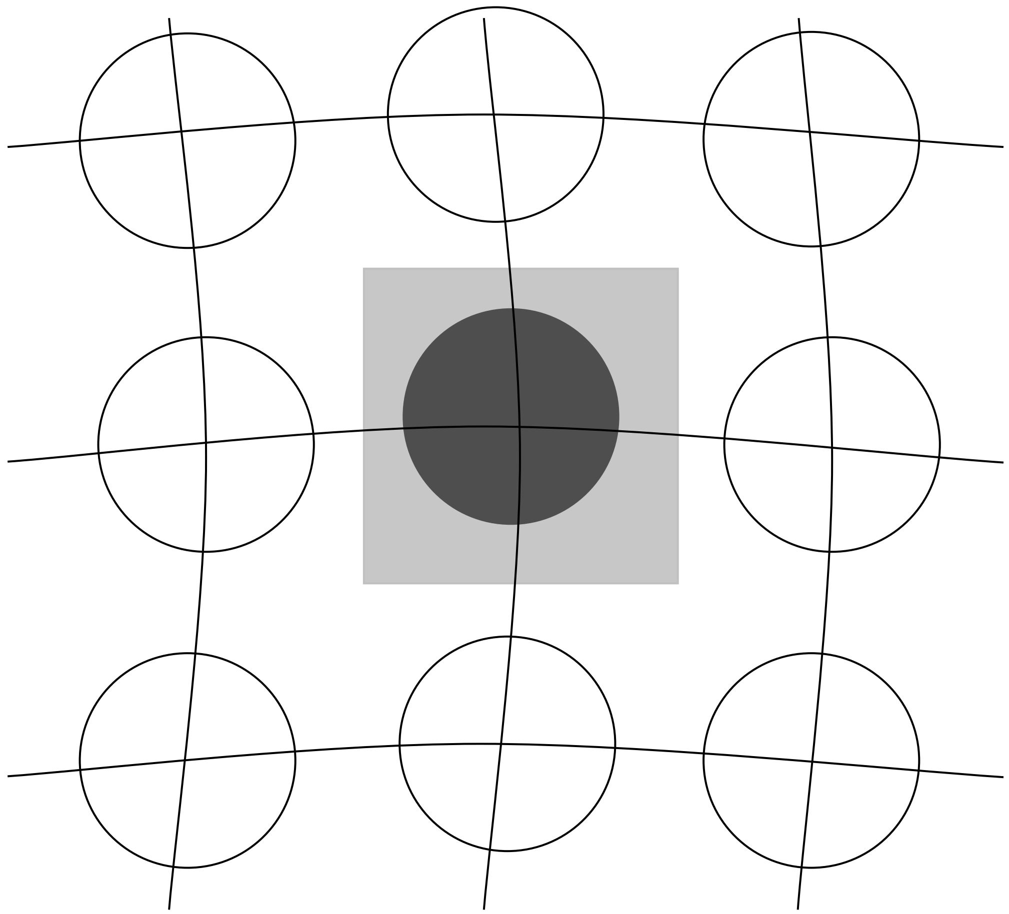
Number of Monte Carlo runs

Mean	Sdev
193.38	22.19
193.41	22.19
193.39	22.19
193.38	22.18
193.37	22.21
193.39	22.20
193.41	22.22
193.40	22.19
193.37	22.20
193.36	22.19
193.39	22.20
0.02	0.01
0.01 %	0.05 %

1000000 runs



Closer and closer



Actual measurements dominate the uncertainty at some point

Method concerns

- Relationship between model and relative uncertainty between points
- Correlation between solid angle determinations
- Extendable to other integrated quantities
- Effect on the industry

Thank you

Questions?